MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN4C. Miscellaneous Results.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights parts of the text that are used in the lecture sessions.

Part 1. About the Eigenvalue Problem

Theorem 4C.1. The Eigenvalue Problem for Identity Matrices

Let $\mathbf{I_n} \in \mathbb{R}^{n\times n}$ be an identity matrix. Then, $\mathbf{I_n}$ has:

eigenvalue λ	algebraic multiplicity	eigenspace E_λ	geometric multiplicity	lin. ind. spanning set of eigenvectors
$\lambda = 1$	n	$E_1 = \mathbb{R}^n$	n	$\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$

Theorem 4C.2. The Eigenvalue Problem on Transformations on $\mathbb{R}^2$ relative to a Line

Let the line $L \subset \mathbb{R}^2$ be spanned by $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \mathbf{0}$. Let $\mathbf{w} = \begin{pmatrix} -b \\ a \end{pmatrix}$, a vector orthogonal to $\mathbf{v}$.

We consider two transformations relative to L.

(a) Let $\mathbf{A_R} \in \mathbb{R}^{2\times 2}$ represent the reflection on $\mathbb{R}^2$ about the line L. Then, $\mathbf{A_R}$ has two real eigenvalues:

eigenvalue λ	algebraic multiplicity	eigenspace E_λ	geometric multiplicity	lin. ind. spanning set of eigenvectors
$\lambda_1 = 1$	1	$E_1 = \text{span}\{\mathbf{v}\}$	1	$\{\mathbf{v}\}$
$\lambda_2 = -1$	1	$E_{-1} = \text{span}\{\mathbf{w}\}$	1	$\{\mathbf{w}\}$

(b) Let $\mathbf{A_O} \in \mathbb{R}^{2\times 2}$ represent the orthogonal projection on $\mathbb{R}^2$ onto the line L. Then, $\mathbf{A_O}$ has two real eigenvalues satisfying:

eigenvalue λ	algebraic multiplicity	eigenspace E_λ	geometric multiplicity	lin. ind. spanning set of eigenvectors
$\lambda_1 = 1$	1	$E_1 = \text{span}\{\mathbf{v}\}$	1	$\{\mathbf{v}\}$
$\lambda_2 = 0$	1	$E_0 = \text{span}\{\mathbf{w}\}$	1	$\{\mathbf{w}\}$

Theorem 4C.3. The Eigenvalue Problem on Rotations on $\mathbb{R}^2$

Let $\mathbf{A}_{\text{rot}} \in \mathbb{R}^2$ represent a rotation on $\mathbb{R}^2$ by $\theta \in \mathbb{R}$ radians, counterclockwise from the positive x-axis. Consider the following cases:

(a) Assume $\theta = 2\pi k$ for some $k \in \mathbb{Z}$. Then, $\mathbf{A}_{\text{rot}} = \mathbf{I_2}$.

(b) Assume $\theta = \pi + 2\pi k$ for some $k \in \mathbb{Z}$. Then, $\mathbf{A}_{\text{rot}}$ has one eigenvalue:

eigenvalue λ	algebraic multiplicity	eigenspace E_λ	geometric multiplicity	lin. ind. spanning set of eigenvectors
$\lambda = -1$	1	$E_{-1} = \mathbb{R}^2$	1	$\{\mathbf{e_1}, \mathbf{e_2}\}$

(c) Assume θ does not satisfy cases (a) and (b). Then, $\mathbf{A}_{\text{rot}}$ has no real eigenvalues. It has the characteristic polynomial $\lambda^2 - 2\cos\theta + 1$ with roots/eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(2\cos\theta \pm \sqrt{4\cos^2\theta - 4}\right) = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

Observe that for cases (a) and (b), $\cos\theta = \pm 1$ and $\cos^2\theta - 1 = 0$.

Theorem 4C.4. The Eigenvalue Problem on Transformations on $\mathbb{R}^3$ relative to a Plane

Let $P \subset \mathbb{R}^3$ be a plane with normal vector $\mathbf{N}$ intersecting the origin. Let $\{\mathbf{d_1}, \mathbf{d_2}\}$ be some set of vectors that span P. Observe that $\mathbf{N}$ is orthogonal to both $\mathbf{d_1}$ and $\mathbf{d_2}$. Then,

(a) $\mathbf{A_R} \in \mathbb{R}^{3\times 3}$ represent the reflection on $\mathbb{R}^3$ about the plane P. Then, $\mathbf{A_R}$ has:

eigenvalue λ	algebraic multiplicity	eigenspace E_λ	geometric multiplicity	lin. ind. spanning set of eigenvectors
$\lambda_1 = 1$	2	$E_1 = P$	2	$\{\mathbf{d_1}, \mathbf{d_2}\}$
$\lambda_2 = -1$	1	$E_{-1} = \text{span}\{\mathbf{N}\}$	1	$\{\mathbf{N}\}$

(b) Let $\mathbf{A_R} \in \mathbb{R}^{2\times 2}$ represent the orthogonal projection on $\mathbb{R}^2$ onto the line L. Then, $\mathbf{A_R}$ has:

eigenvalue λ	algebraic multiplicity	eigenspace E_λ	geometric multiplicity	lin. ind. spanning set of eigenvectors
$\lambda_1 = 1$	2	$E_1 = P$	2	$\{\mathbf{d_1}, \mathbf{d_2}\}$
$\lambda_2 = 0$	1	$E_0 = \text{span}\{\mathbf{N}\}$	1	$\{\mathbf{N}\}$

Part 2. Inverse Matrices by Cofactor Expansion

Definition 4C.5. Matrix Minors and Cofactors

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Denote $\mathbf{A}[i,j] \in \mathbb{R}^{(n-1)\times(n-1)}$ as the matrix generated by removing the i^{th} row and the j^{th} column of $\mathbf{A}$.

(a) The $(i,j)^{\text{th}}$ **first minor** $M[i,j] \in \mathbb{R}$ of $\mathbf{A}$ is given by $\det(\mathbf{A}[i,j])$.

(b) The $(i,j)^{\text{th}}$ **cofactor** $C[i,j] \in \mathbb{R}$ of $\mathbf{A}$ is given by $(-1)^{i+j}M[i,j] = (-1)^{i+j}\det(\mathbf{A}[i,j])$.

(c) The **cofactor matrix** $\mathbf{C} \in \mathbb{R}^{n\times n}$ of $\mathbf{A}$ is given component-wise by $\mathbf{C}_{i,j} = C[i,j]$.

Definition 4C.6. Determinant in terms of Cofactors

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Denote $\mathbf{A}[i,j] \in \mathbb{R}^{(n-1)\times(n-1)}$ as the matrix generated by removing the i^{th} row and the j^{th} column of $\mathbf{A}$. Then, the determinant, $\det(\mathbf{A})$, of $\mathbf{A}$ can be calculated using:

$$\text{For fixed } i\text{, expand along } \mathsf{row}_i(\mathbf{B}): \quad \det(\mathbf{A}) = \sum_{j=1}^{n}(-1)^{i+j}(\mathbf{A})_{i,j}\det(\mathbf{B}[i,j]) = \sum_{j=1}^{n}(\mathbf{A})_{i,j}C[i,j]$$

$$\text{For fixed } j\text{, expand along } \mathsf{col}_j(\mathbf{B}): \quad \det(\mathbf{A}) = \sum_{i=1}^{n}(-1)^{i+j}(\mathbf{A})_{i,j}\det(\mathbf{B}[i,j]) = \sum_{i=1}^{n}(\mathbf{A})_{i,j}C[i,j]$$

where $C[i,j]$ is the $(i,j)^{\text{th}}$ cofactor of $\mathbf{A}$.

Theorem 4C.7. Inverse Matrices by Cofactor Expansion
Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be invertible. Then, the inverse matrix $\mathbf{A}^{-1}$ can be calculated using

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^\top$$

where $\mathbf{C}$ is the cofactor matrix of $\mathbf{A}$.